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DEGRADATION IN GAIN FOR A FREE ELECTRON LASER AMPLIFIER DUE TO ELECTRON MOMENTUM SPREAD*

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A finite spread in axial momentum for the electron beam in a free electron laser amplifier is shown to decrease the small-signal gain. For millimeter and sub-millimeter wave amplifiers, where exponential growth dominates the gain, it is shown that the gain is approximately 3 db below that for a cold beam if the relative momentum spread $(\Delta u/u)_{1/2} = (G_0/248)^{1/2}(\lambda_0/L)$, where $G_0 >> 1$ is the gain in db for the cold-beam case, λ_0 is the magnetic wiggler period, and L is the amplifier length. Exact numerical examples are given for representative FEL amplifiers at 35 and 550 GHz.

Key words: free electron laser, amplifier, electron momentum spread.

Most theoretical work concerning amplification of radiation in free electron lasers (FELs) deals of necessity with idealized models. One idealization widely employed involves the neglect of finite momentum spread of the electron beam. The underlying mechanism for small-signal amplification involves axial synchronization in propagation velocity between one of the allowed modes of radiation supported by the beam, and the beam itself. Thus when a spread in axial beam momentum is present, a mixing-in-phase can be expected to degrade the amplification which would otherwise be predicted for a cold beam. Prior workers (1,2) have taken note of this fact and have provided estimates of the

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effect of momentum spread. This paper presents an exact analytical model to account for finite momentum spread for a particular distribution function. When exponential growth dominates the gain, a simple approximate formula is derived to estimate the loss in gain due to the momentum spread. Exact numerical examples are also given for representative FEL amplifiers at 35 and 550 GHz.

The basic FEL model adopted here is identical to that treated by Bernstein and Hirshfield (B-H) (3). That work gave an exact small-signal solution of the Vlasov-Maxwell equations for the steady-state evolution of the co-propagating disturbance which grows in space on a relativistic electron beam passing along the axis of a helical magnetic wiggler. The B-H theory was derived for a beam of arbitrary momentum distribution in a wiggler of arbitrary strength, but the solutions presented were for the case of a cold beam, viz.,

$$f_o(\alpha,\beta,u) = N_o\delta(\alpha)\delta(\beta)\delta(u-U) \ . \qquad (1)$$

Here α and β are the two transverse components of canonical angular momentum $U_x - eA_x/mc^2$ and $U_y - eA_y/mc^2$, A_x and A_y are the components of the wiggler's vector potential, U_x and U_y are the transverse components of translational momentum, and $U = (\gamma^2 - 1)^{1/2}$ is the total momentum as related to the relativistic energy factor . (All momenta are normalized to mc.) Eq. (1) thus describes a beam which, prior to entering the wiggler, contains electrons possessing both zero transverse momentum and unique axial momentum U.

As mentioned above, an important source of degraded amplification is the finite spread of axial momentum on the electron beam. In the work reported here, we choose the simplest distribution capable of describing such a spread, viz.,

$$f_o(\alpha,\beta,u) = N_o\delta(\alpha)\delta(\beta)\left[\frac{H(u-U_1) - H(u-U_2)}{\Delta U}\right] , \qquad (2)$$

where $H(x) = 1$ for $x > 0$, $H(x) = 0$ for $x < 0$, and $\Delta U = U_2 - U_1 > 0$. This distribution can of course not be realized in nature [in the same sense that the distribution given by Eq. (1) cannot]. It may, however, not be a bad approximation for certain accelerators (except for the sharp edges); but its utility here is that it enables an analytic form to be derived for the governing dispersion relation.

The goal of the present work is identical to that in B-H, namely to calculate the power gain G (in db) for a single pass of electromagnetic radiation along a FEL amplifier of length L.

$$10^{G/10} = a_2(L)a_2^*(L) - 1 \tag{3}$$

Here $a_2(L)$ is the dimensionless wave electric field at the amplifier output, normalized to unity at the input. The subscript "2" labels one of the three polarizations permitted, namely that which twists in space a quarter-period behind the wiggler's vector potential. [Eqs. (35) and (37) in B-H give the other two polarizations.]

The wave amplitude $a_2(L)$ is a superposition of several co-propagating normal modes, each with its wavenumber k_j, viz.,

$$a_2(L) = \sum_j \frac{B(k_j)}{R'(k_j)} \exp(ik_jL) \tag{4}$$

The relative mode amplitudes $B(k_j)/R'(k_j)$ are prescribed once boundary conditions are set. $R(k_j) = 0$ is the dispersion relation for the system which determines the $k_j(\omega)$, assuming $R^{-1}(k)$ to have simple poles. For the cold beam case $R(k_j)$ is a sixth-order polynomial.

$$R(x) = [(x-\mu)^2 - \delta^2(1+\xi^2)][(x+x_o)^2 - b^2][(x-x_o)^2 - b^2] + \xi^2\delta^2(x^2-b^2)(x^2+x_o^2-b^2) , \tag{5}$$

where $x = kc/\omega$, $x_o = k_oc/\omega$; $\delta = (\omega_p/\omega)(U/\gamma U_z^3)^{1/2}$, $b = (1-U_z^2\delta^2)^{1/2}$, $\mu = \gamma/U_z$, $U_z = (U^2-\xi^2)^{1/2}$, and $\xi = - eB_o/mc^2k_o$. The wiggler field strength and wavenumber are B_o and k_o. This equation has been obtained as well by Sprangle (1), and related forms have been derived and discussed by Kroll and McMullin (2) and by Kwan, Dawson, and Lin (4). When $\delta << x_o << 1$, a reduced form of Eq. (5) is a good approximation, namely

$$R(x) \simeq [(x-\mu)^2 - \delta^2(1+\xi^2)][x - (b+x_o)] + \frac{1}{2}\xi^2\delta^2 x_o \quad . \qquad (6)$$

For $k_o/k \lesssim (1+\xi^2)/2\gamma^2$ the maximum growth occurs near $b+x_o - \mu = (\delta^2\xi^2 x_o/2)^{1/2}$. To requisite accuracy the roots are

$$\begin{aligned} x_1 &= \mu + (\delta^2\xi^2 x_o/2)^{1/3}\exp(-\pi i/3) \\ x_2 &= x_1^* \\ x_3 &= \mu - (\delta^2\xi^2 x_o/2)^{1/3} \quad . \end{aligned} \qquad (7)$$

These roots are of use in scaling estimates when exponential gain is dominant. Exact numerical evaluations given in B-H show, however, that Eq. (7) cannot be used to determine the entire gain spectrum.

When Eq. (2) is employed as the distribution function all the momentum-space integrals in the Vlasov formulation can be expressed analytically. We then find

$$\begin{aligned} R(x) = {} & [(x-\mu_1)(x-\mu_2) - \delta'^2(1+\xi^2)][(x+x_o)^2 - b'^2] \\ & \times [(x-x_o)^2 - b'^2] + \xi^2\delta'^2(x^2-d^2)(x^2+x_o^2-b'^2) \quad , \end{aligned} \qquad (8)$$

where

$$\delta'^2 = -\frac{\omega_p^2}{\omega^2}\,\frac{1}{1+\xi^2}\,\frac{\Delta\mu}{\Delta U} \quad ;$$

$$b'^2 = 1 - \frac{\omega_p^2}{\omega^2}\,\frac{1}{\Delta U}\,\ell n\left(\frac{\gamma_2 + U_{z2}}{\gamma_1 + U_{z1}}\right) \quad ;$$

$$d^2 = 1 - \frac{1}{\Delta U}\,\frac{\omega_p^2}{\omega^2}\left[\Delta\mu - \frac{1+\xi^2}{\Delta\mu}\left(\frac{1}{U_{z1}} - \frac{1}{U_{z2}}\right)^2\right] \quad ;$$

$\Delta\mu = \mu_2 - \mu_1 < 0$, $\gamma_{1,2}^2 = 1 + U_{1,2}^2$, $U_{z1,2}^2 = U_{1,2}^2 - \xi^2$, and $\mu_{1,2} = \gamma_{1,2}/U_{z1,2}$.

When $\Delta U/U << 1$, $\Delta\mu \simeq - U\Delta U(1 + \xi^2)\gamma U_z^3$, $\delta' \simeq \delta$, and $b' \simeq d \simeq b$. Thus the only effect of finite momentum spread in this limit is in the factor $(x - \mu_1)(x - \mu_2) = (x - \bar{\mu})^2 - (\Delta\mu/2)^2$ in the first bracket in Eq. (8), where $\bar{\mu} = (\mu_1 + \mu_2)/2$. The close similarity between Eqs. (8) and (5), and the simplicity of the former, make determination of the roots k_j a routine matter. This simplicity is not enjoyed when the momentum spread is described by functions $f_o(\alpha,\beta, U)$ with non-zero values of $\partial f_o/\partial U$ in a finite interval, because of wave-particle resonance effects.

As for the cold-beam case, where $\delta' << x_o << 1$, Eq. (8) may be reduced to the approximate form

$$R(x) \simeq [(x-\bar{\mu})^2 - (\Delta\mu/2)^2][x - (b'+x_o)] + \xi^2\delta'^2 x_o/2 = 0. \quad (9)$$

If $(\Delta\mu/2)^2 << 3(\xi^2\delta'^2 x_o/2)^{1/3}$, the roots of Eq. (9) near $b' + x_o - \bar{\mu} = (\xi^2\delta'^2 x_o/2)^{1/2}$ are approximately

$$\begin{aligned} x_1 &= \bar{\mu} + (\xi^2\delta'^2 x_o/2)^{1/3}\exp(i\pi/3) \\ &\qquad + \tfrac{1}{3}(\Delta\mu/2)^2(\xi^2\delta'^2 x_o/2)^{-1/3}\exp(-i\pi/3) \\ x_2 &= x_1^* \\ x_3 &= \bar{\mu} - (\xi^2\delta'^2 x_o/2)^{1/3} - \tfrac{1}{3}(\Delta\mu/2)^2(\xi^2\delta'^2 x_o/2)^{-1/3} . \end{aligned} \quad (10)$$

Thus the spatial growth constant $\mathrm{Im}x_1$ is seen to decrease on account of momentum spread as

$$\mathrm{Im}x_1 \simeq \frac{\sqrt{3}}{2}\left(\xi^2\delta'^2 x_o/2\right)^{1/3}\left[1 - \frac{1}{3}\left(\frac{\Delta\mu}{2}\right)^2\left(\xi^2\delta'^2 x_o/2\right)^{-2/3}\right]. \quad (11)$$

For pure exponential gain, i.e. excluding the 15.6 db input coupling loss (see B-H), one has

$$G = 54.58(L/\lambda)\,\mathrm{Im}x_1 \quad \mathrm{db} \quad (12)$$

where λ is the radiation wavelength. From Eq. (11) we can write $G = G_o - G_1$, where G_o is the gain with no momentum spread, and G_1 is the small decrease due to the spread

$$G_o = 54.58(L/\lambda)\frac{\sqrt{3}}{2}(\xi^2\delta^2 x_o/2)^{1/3} \text{ db} \quad . \qquad (13)$$

For $\xi = 0.47$, $\lambda = 4.9$, $x_o = 2.73 \times 10^{-2}$, $\delta^2 = 3.80 \times 10^{-6}$, and $L/\lambda = 367$ (corresponding to a representative FEL amplifier to be discussed below), Eq. (13) gives $G_o = 39.1$ db. [If one subtracts the 15.6 db input coupling loss, the actual gain would be 23.5 db (at a wavelength of 560 μm).] Now

$$G_1 = \frac{54.58}{8\sqrt{3}} \frac{L}{\lambda} (\Delta\mu)^2 (\frac{1}{2} \xi^2 \delta^2 x_o)^{-1/3} \text{ db} \quad . \qquad (14)$$

Substituting from Eq. (13) gives the value of $\Delta\mu$ which would bring about a gain loss G_1

$$(\Delta\mu)^2 = 5.37 \times 10^{-3} G_o G_1 (\lambda/L)^2 \quad . \qquad (15)$$

For the example cited above with $L/\lambda = 367$ we find $\Delta\mu = 2.16 \times 10^{-3}$ for $G_o = 39.1$ db and $G_1 = 3$ db, i.e. for a factor-of-two decrease in power amplification. This corresponds to a relative momentum spread $\Delta U/U = |\Delta\mu|[\gamma U_z^3/U^2(1 + \xi^2)]$ of 0.041.

Equation (10) also suggests that the frequency at which gain has its peak value will decrease as momentum spread increases.

Exact numerical evaluations for small-signal gain G have been carried out using the full dispersion relation [Eq. (8)], and with amplitudes [see Eq. (4)] appropriate to a perfectly matched amplifier output. One example is for a mm-wave amplifier employing an electron beam typical of that produced by a small Febetron accelerator, with $\gamma = 1.78$, $J = 100$ A/cm^2, $\lambda_o = 3.6$ cm, $\xi = 0.2$, and $L = 36$ cm. Gain curves are shown in Fig. 1 for zero momentum spread, and for finite momentum spreads between 5 and 20%. Gain is seen to fall by one-half for $\Delta U/U \simeq 0.15$, and the frequency for peak gain drops by about 6%. A second example is for a sub-mm wave amplifier employing a beam typical of the VEBA accelerator at Naval Research Laboratory, with $\gamma = 4.9$, $J = 6$ kA/cm^2, $\lambda_o = 2.0$ cm, $\xi = 0.47$, and $L = 20$ cm. For this case the computed gain curves are shown in Fig. 2, again for zero momentum spread and for spreads between 5 and 20%.

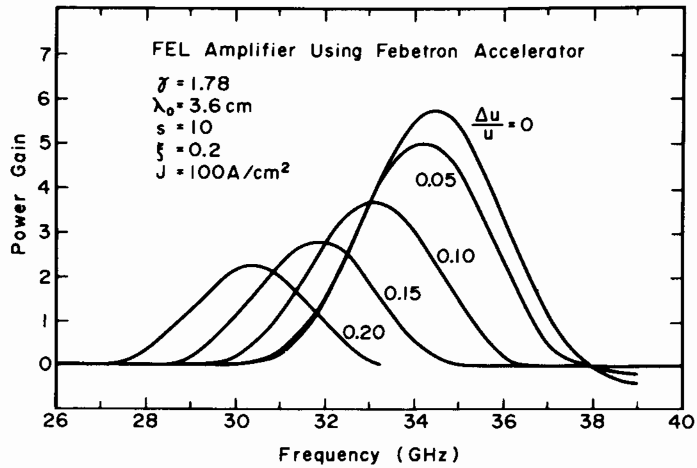

Figure 1. Gain curves for a FEL using a 400 kV electron beam, for electron momentum spread between 0 and 20%.

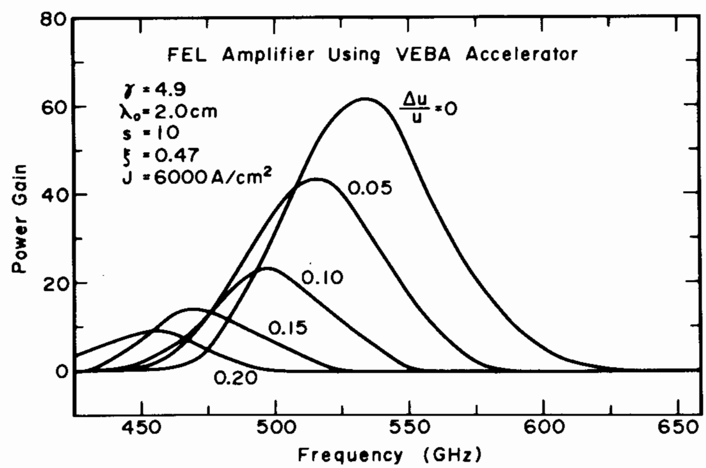

Figure 2. Gain curves for a FEL using a 2.0 MV electron beam, for electron momentum spread between 0 and 20%.

Comparisons between the exact results (Fig. 2) and the approximate predictions [Eqs. (12-15)] are instructive. The peak gain for the cold beam is 17.8 db (i.e. 60×) compared with the approximate value of 23.5 db. The gain drops by half to 14.8 db (i.e. 30×) for $\Delta U/U$ somewhat greater than 10%; our approximate result is 4.1%. These comparisons for the example presented in Fig. 1 are not meaningful since the peak gain G_o is less than 7.8 db (6×).

Finally, we point out the scaling laws suggested by Eqs. (12-15), valid for high gain devices where exponential growth dominates. For negligible momentum spread,

$$G_o \sim J^{1/3} L \xi^{2/3} \lambda^{-2/3} \lambda_o^{-1/3} \text{ db}$$

or equivalently (16)

$$G_o \sim J^{1/3} (L/\lambda_o) \xi^{2/3} \gamma^{4/3} \text{ db} \quad .$$

For the gain decrease $G_1 << G_o$ due to finite momentum spread we have

$$G_1 G_o \sim (\Delta U/U)^2 (L/\lambda_o)^2 \ (\text{db})^2 \quad . \tag{17}$$

Eq. (17) indicates that high gain short amplifiers are less susceptible to gain degradation due to momentum spread, than are low gain long amplifiers. This scaling is independent of λ and γ provided G_o is high. For $G_1 = 3$ db, the numerical value for Eq. (17) gives $(\Delta U/U)_{1/2} = (G_o/248)^{1/2} (\lambda_o/L)$, where $(\Delta U/U)_{1/2}$ is the relative momentum spread for a factor-of-two decrease in gain. Gain degradation for long-wiggler FELs operating in the collective regime can be expected to be serious unless $\Delta U/U << 1$.

It should be added as a caveat however that momentum spread may not always degrade gain in a FEL. The geometrical optics theory for a FEL amplifier (5) shows that gain may arise from a wave-particle resonance, provided $f_o(\alpha,\beta,u)$ is not symmetric in u about its maximum, and provided $\partial f_o/\partial u$ has the requisite sign at the wave's phase velocity. It is expected that this mechanism would compete with that discussed in the present paper, and could in fact allow substantial gain in the presence of tailored momentum spread.

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